HOMEWORK I (DUE DATE: 23/02/2023)

Exercise 1 (2 point)**.** Solve the following transport equation.

(1) $u_t + 5u_x = e^{3t}$, $u(x,0) = e^{-x^2}$.
(2) $u_t - xu_x = 0$, $u(x,0) = x^3 - 1$.

Exercise 2 (2 point)**.** Give the types and the canonical forms of the following equations, then find the solutions.

(1) $u_{xx} + 5u_{xy} + 6u_{yy} = 0$.
(2) $y^2 u_{xx} - 2yu_{xy} + u_{yy} - u_x + 6y = 0$.

Exercise 3 (2 point)**.**

(1) Show that the function $u = \log(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 \backslash \{(0,0)\}$.
(2) The Laplacian in the planar cartesian coordinate system is defined as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

rewrite the Laplacian in the planar polar coordinate system and verify (1). (Hints: Recall the transformation between the cartesian coordinate system and he planar polar coordinate system is determined by $x = r\cos\theta$, $y = r\sin\theta$.)

Exercise 4 (2 points)**.** Let u be a real $C^1(\mathbb{R}^2)$ solution of the equation

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = -u(x,y),$$

in the closed unit disc $D \subset \mathbb{R}^2$. We assume here that a and b are given C^1 real coefficients with

$$a(x,y)x + b(x,y)y > 0$$

on the unit disc circle. Show that $u \equiv 0$. (Hint: One can show that u^2 can not have a positive maximum.)

Exercise 5 (2 points)**.**

(1) Let u be harmonic in a domain $V \subset \mathbb{R}^2$. Prove

$$|\nabla u(x_0, y_0)| \le \frac{C}{r} \max_{\partial B_r(x_0,y_0)} |u|,$$

where C is a constant.
(2) Let u be harmonic in $\mathbb{R}^2$, and $0 < \gamma < 1$, $C > 0$ such that

$$|u(x,y)| \le C(1 + |x|^2 + |y|^2)^{\frac{\gamma}{2}}, \quad \forall (x,y) \in \mathbb{R}^2.$$

Prove that u is a constant function.